Computing a Condorcet winner of a 1-Euclidean election

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Abstract

In this paper, we are concerned with the problem of deploying public facilities via a 1-Euclidean election under the majority rule. In a 1-Euclidean election, voters and candidates can be mapped into $\mathbb{R}^1$, and each voter's preference is determined by the distances from the voter to the candidates. Specifically, each candidate considered in this work consists of arbitrary k points, and the winner is determined with Condorcet criterion. Given that k is fixed, we show that determining whether a Condorcet winner exists can be done in time linear to the number of voters.

1 Introduction

We start with the definition of the problem. The election considered in this paper consists of three things, voters, candidates, and how a voter prefers a candidate to another. In the *1-Euclidean* election we are concerned with, voters are n points in $\mathbb{R}^1$, and candidates are all subsets of $\mathbb{R}^1$ of size k. Let $d(x, y)$ be the distance between x and y. The distance from a point x to a set Y is defined as

$$\min \{d(x, y) \colon\ y \in Y\},$$

also denoted by $d(x, Y)$. A voter x prefers candidate Y to candidate Z if $d(x, Y) < d(x, Z)$. A *Condorcet winner* is a candidate such that no alternative can please more voters than it does. Our goal is to compute a Condorcet winner of a 1-Euclidean election if one exists, or report the non-existence.

Related work

For $k = 1$, a Condorcet winner always exists and coincides with a *median* [5]. For $k = 1$ and $\mathbb{R}^d$ with $d > 1$, Wu et al. [17] proposed an $O(n^{d-1} \log n)$-time algorithm. Later on, de Berg et al. [8] revised the time complexity to $O(n \log n)$. Respecting Condorcet winners for $k > 1$, to our understanding, related results have been developed only in $\mathbb{R}^1$. Barberà and Beviá [3, 4] gave some properties of a Condorcet winner consisting of k points, namely the *internal consistency*, *Pareto feasibility*, and *Nash stability.* Hajduková [11] then developed an algorithm that verifies if a given decision is a Condorcet winner.

There are several results regarding the computation of a Condorcet winner on graphs. We refer the reader to [2, 12, 13, 16]. Results regarding the structure of voters' preferences are also widely developed [6, 9, 14, 15]. See [10] for a brief survey. The reason why people pay attention to this kind of elections is that such elections have a natural interpretation, like locating *facilities* into the space to meet voters' demands. In this paper, we also call the k points that constitute a candidate the facilities.

In the rest of the paper, we first summarize some preliminary results in Section 2. Then, in Sections 3 and 4 we reduce the solution space so that an enumerative procedure is applicable. The analysis of the time complexity is given in Section 5. Omitted proofs are given in the appendix.

2 Preliminaries

Let $[n]$ be the set of integers $\{1, \ldots, n\}$, and let S be the set of voters. We assume $S = [n]$. For $i \in S$, the point that corresponds to i is denoted by p_i. We assume that $i < j$ implies $p_i < p_j$. A subset of voters is called a *community*. Let $P_S = \{p_i\colon i \in S\}$, the *preference profile*, by which one can determine how a voter prefers one candidate to another. An instance is a triple (S, k, P_S), where k is the number of facilities that constitute a candidate.

For the instance (S, k, P_S), an *S/k-decision* $\big((x_h, S_h)\big)_{h=1}^{k}$ is a k-tuple of pairs, where $x_h \in \mathbb{R}$ with $x_1 < \cdots < x_k$ and $(S_1, \ldots, S_k)$ is a partition of S. We use the term "decision" if there is no danger of misinterpretation. For a decision $d = ((x_h, S_h))_{h=1}^k$, voter i is *assigned to* x_j if $i \in S_j$, denoted by $x_j = x(i, d)$. We refer to $(x_1, \ldots, x_k)$ and $(S_1, \ldots, S_k)$ as d_L and d_A, respectively. For notational succinctness, d_L and d_A are also used as the sets with the corresponding elements.

For two points $x, y \in \mathbb{R}^1$, voter i prefers x to y, denoted by $y \prec_i x$, if $|x - p_i| < |y - p_i|$. Analogously, for two S/k-decisions d and d', voter i prefers d' to d, denoted by $d \prec_i d'$, if $x(i, d) \prec_i x(i, d')$.

Definition 1 (Condorcet winner)**.** *Given an instance (S, k, P_S), an S/k-decision d^* is a Condorcet winner if there is no S/k-decision d such that*

$$|\{i \in S\colon\ d \prec_i d^*\}| < |\{i \in S\colon\ d^* \prec_i d\}|.$$

Note that the definition relaxes the one given in the beginning of Section 1 since the partition of voters does not depend on the facilities. With the *envy-freeness* defined below, the sets of Condorcet winner of both formulations are identical.

An S/k-decision $d = ((x_i, S_i))_{i=1}^k$ is *envy-free* if for $i \in S$ and $j \in [k]$, $x_j \preceq_i x(i, d)$. Decision d is *internally consistent* if for $i \in [k]$, (x_i, S_i) is a Condorcet winner of $(S_i, 1, P_{S_i})$. In other words, a decision is internally consistent if x_i coincides with a median of P_{S_i}, for $i \in [k]$.

Proposition 1 (Barberà and Beviá [3, 4])**.** *Given an instance (S, k, P_S) and a decision $d = ((x_h, S_h))_{h=1}^k$, if d is a Condorcet winner, then d is envy-free and internally consistent.*

Proposition 1 gives necessary conditions for being a Condorcet winner. To determine whether a given decision is a Condorcet winner, Hajduková gave the notion of *simple rival*, which makes the verification feasible. Given an instance (S, k, P_S), let d and d' be two S/k-decisions such that $d_L = (x_1, \cdots, x_k)$ and $d'_L = (x'_1, \cdots, x'_k)$. Let $\Delta(d, d') = \{j \in [k]\colon\ x_j \neq x'_j\}$. The decision d' is a *potential rival* of d if

- $\Delta(d, d') \neq \emptyset$;
- for $j_1 < j_2 < j_3$, $\{j_1, j_3\} \subseteq \Delta(d, d')$ implies $j_2 \in \Delta(d, d')$;
- for $i \in \Delta(d, d')$, either $x_i < x'_i$ or $x'_i < x_i$.

If d' further satisfies

$$\big|\{i \in S\colon\ d' \prec_i d\}\big| < \big|\{i \in S\colon\ d \prec_i d'\}\big|,$$

then d' is a simple rival of d. Figure 1 gives an example.

Proposition 2 (Hajduková [11])**.** *For an instance (S, k, P_S), an S/k-decision d is a Condorcet winner if and only if d is envy-free and has no simple rival.*

Note that a decision with no simple rival may not be envy-free (Figure 2). Hajduková's verification algorithm was developed based on Proposition 2. The envy-freeness can be verified in a straightforward manner, while determining the existence of a simple rival needs a careful counting on the gain and loss of the votes, as shown in Section 3.

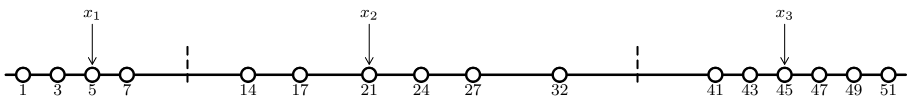

Figure 1: An envy-free decision d with $d_L = (5, 21, 45)$. Decision d has a simple rival which is an envy-free decision d' with $d'_L = (21, 42, 47)$.

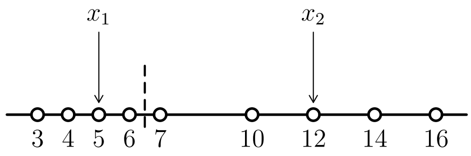

Figure 2: A non-envy-free decision d with no simple rival. Decision d is not a Condorcet winner since more voters prefer decision $d' = \big((5, \{3, 4, 5, 6, 7\}), (12, \{10, 12, 14, 16\}\big)$ to d.

3 The score of a decision

Consider the instance (S, k, P_S). For two S/k-decisions d and d', let

$$N(d', d) = |\{i \in S \colon d \prec_i d'\}| .$$

The *margin* of d' with respect to d is defined as

$$N_d(d') = N(d', d) - N(d, d').$$

Let $d_L = (x_1, \cdots, x_k)$. Assume that $x_0 = -\infty$ and $x_{k+1} = \infty$. For $0 \le i \le k$, let

$$N_d(d')|_i = |\{j \in S \colon p_j \in (x_i, x_{i+1}), d \prec_j d'\}| - |\{j \in S \colon p_j \in (x_i, x_{i+1}), d' \prec_j d\}| .$$

Then we have

$$N_d(d') = \sum_{i=0}^{k} N_d(d')|_i - |P_S \cap (d_L \setminus d'_L)|. \tag{1}$$

Assume that d^* is an S/k-decision that maximizes $N_d(\cdot)$. Obviously, d is a Condorcet winner if and only if $N_d(d^*) \le 0$.

Lemma 1. *For $x < x' < z' < z$, the following statements are equivalent.*

- $|x' - z'| \le |x - z|/2$.
- *There is a point y such that any point w in (x', z') satisfies $x \prec_w y$ and $z \prec_w y$.*

Proof. Omitted. □

With Lemma 1, we may compute $N_d(d^*)$ as follows. Note that by deploying the two facilities at $x_i + \epsilon$ and $x_{i+1} - \epsilon$, each voter in the interval prefers d^* to d.

Observation 1. *There are at most two facilities of d^* in the interval (x_i, x_{i+1}), for $0 \le i \le k$.* □

For an instance (S, k, P_S), we define the following *scoring functions*, c, f, g^+, and g^-.

$$\begin{aligned}
c(x,y) &= |P_S \cap (x,y)| \\
f(x,z) &= \max\left\{ c\left(\frac{x+y}{2}, \frac{y+z}{2}\right) \,:\, y \in (x,z) \right\} \\
g^+(x,z) &= \max\left\{ c\left(\frac{x+y}{2}, \frac{y+z}{2}\right) - c\left(x, \frac{x+y}{2}\right) \,:\, y \in (x,z) \right\} \\
g^-(x,z) &= \max\left\{ c\left(\frac{x+y}{2}, \frac{y+z}{2}\right) - c\left(\frac{y+z}{2}, z\right) \,:\, y \in (x,z) \right\}.
\end{aligned}$$

Since S is finite, the above functions are well-defined. With Observation 1, $N_d(d^*)|_i$ is determined as follows.

Proposition 3. *Given an instance (S, k, P_S) and an S/k-decision d with $d_L = (x_1, \ldots, x_k)$, let d^* be an S/k-decision that maximizes $N_d(\cdot)$. For $0 \le i \le k$, if $|\{x_i, x_{i+1}\} \cap d^*_L| = 0$, then*

$$N_d(d^*)|_i = \begin{cases} -c(x_i, x_{i+1}), & \text{if } |d^*_L \cap (x_i, x_{i+1})| = 0 \\ 2f(x_i, x_{i+1}) - c(x_i, x_{i+1}), & \text{if } |d^*_L \cap (x_i, x_{i+1})| = 1 \\ c(x_i, x_{i+1}), & \text{if } |d^*_L \cap (x_i, x_{i+1})| = 2. \end{cases}$$

Proposition 4. *Given an instance (S, k, P_S) and an S/k-decision $d = ((x_i, S_i))_{i=1}^k$, let d^* be an S/k-decision that maximizes $N_d(\cdot)$. For $0 \le i \le k$, if $|\{x_i, x_{i+1}\} \cap d^*_L| = 1$, then*

$$N_d(d^*)|_i = \begin{cases} -n_i^+ \text{ or } -n_{i+1}^-, & \text{if } |d^*_L \cap (x_i, x_{i+1})| = 0 \\ g^+(x_i, x_{i+1}) \text{ or } g^-(x_i, x_{i+1}), & \text{if } |d^*_L \cap (x_i, x_{i+1})| = 1 \\ c(x_i, x_{i+1}), & \text{if } |d^*_L \cap (x_i, x_{i+1})| = 2, \end{cases}$$

where $n_i^- = |\{j \in S_i \colon p_j < x_i\}|$ and $n_i^+ = |\{j \in S_i \colon p_j > x_i\}|$.

Proposition 5. *Given an instance (S, k, P_S) and an S/k-decision d with $d_L = (x_1, \ldots, x_k)$, let d^* be an S/k-decision that maximizes $N_d(\cdot)$. For $0 \le i \le k$, if $|\{x_i, x_{i+1}\} \cap d^*_L| = 2$, then*

$$N_d(d^*)|_i = \begin{cases} 0, & \text{if } |d^*_L \cap (x_i, x_{i+1})| = 0 \\ f(x_i, x_{i+1}), & \text{if } |d^*_L \cap (x_i, x_{i+1})| = 1 \\ c(x_i, x_{i+1}), & \text{if } |d^*_L \cap (x_i, x_{i+1})| = 2. \end{cases}$$

Propositions 3, 4, and 5 enable us to compute the maximum of $N_d(\cdot)$ by dynamic programming, as shown in Section 5. To find a Condorcet winner, we reduce the number of decisions to be tested. An essential observation is derived from the scoring functions.

Observation 2. *Given that x is fixed, $f(x, z)$ is nondecreasing on z. Conversely, given x and $f(x, z) = \tau$, z is bounded above depending on x and τ.* □

4 Bounding the position of a facility

To efficiently verify whether a decision is Condorcet, Hajduková further gave some necessary conditions. For a Condorcet winner $d = ((x_i, S_i))_{i=1}^k$ of an instance (S, k, P_S), d satisfies

- $\forall_{i,j \in [k]} \, ||S_i| - |S_j|| \le 2$.

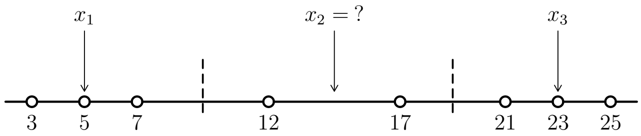

Figure 3: A Condorcet winner of instance $([8], 3, \{3, 5, 7, 12, 17, 21, 23, 25\})$. The decision d with $d_A = (\{1, 2, 3\}, \{4, 5\}, \{6, 7, 8\})$ and $d_L = (x_1, x_2, x_3)$ is a Condorcet winner. As shown in Section 4, $12 < x_2 < 17$. Since neither $x_2 \neq 12$ nor $x_2 \neq 17$, x_2 is singular.

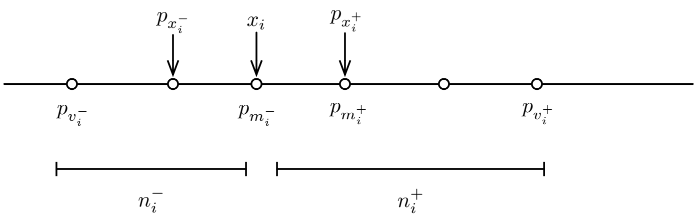

Figure 4: A community S_i. In this example, $n_i^- = 2$ and $n_i^+ = 3$

- $\forall_{i \in [k]}\ |S_i| \neq \min\{|S_j|\colon\ j \in [k]\} \implies x_i \in P_S$.

In the remainder of this section, we assume that the decisions under consideration satisfy the above two conditions. Along with the internal consistency, we call such decisions *regular* decisions. For a Condorcet winner, the property of being regular guarantees that facilities coincide with some voters, except those belonging to the communities whose size is even and minimum. We call such a community S_h *singular*, i.e. $|S_h| = \min\{\,|T|\colon\ T \in d_A\,\}$ and $|S_h|$ is even. The facility x_h is referred to as a singular facility. Note that it is possible for a Condorcet winner to have singular facilities. See Figure 3 for an example.

Below are some notations, illustrated in Figure 4. Given a decision $d = ((x_i, S_i))_{i=1}^k$, we denote the median of S_i by $\mathrm{med}(S_i)$, and for $i \in [k]$ we define the following.

- v_i^+ and v_i^- are the minimal and maximal element of S_i, respectively.
- $n_i^- = |\{j \in S_i\colon\ p_j < x_i\}|$ and $n_i^+ = |\{j \in S_i\colon\ p_j > x_i\}|$.
- $x_i^- = \max\{j \in S_i\colon\ p_j < x_i\}$ and $x_i^+ = \min\{j \in S_i\colon\ p_j > x_i\}$.
- $m_i^- = \max\{j \in S_i\colon\ j \leq \mathrm{med}(S_i)\}$ and $m_i^+ = \min\{j \in S_i\colon\ j \geq \mathrm{med}(S_i)\}$.

Let S_h be a singular community. If $h = 1$, then moving x_h to $p_{m_h^+}$ keeps the property of being a Condorcet winner.

Lemma 2. *Let $d = ((x_i, S_i))_{i=1}^k$ be a Condorcet winner of (S, k, P_S), where $p_{m_1^-} < x_1 < p_{m_1^+}$. If $d' = ((x'_i, S_i))_{i=1}^k$ with*

$$x'_i = \begin{cases} p_{m_1^+}, & \text{if } i = 1 \\ x_i, & \text{otherwise,} \end{cases}$$

then d' is a Condorcet winner of (S, k, P_S).

Proof. Suppose to the contrary that d' is not a Condorcet winner. First, d' is envy-free since otherwise d is not a Condorcet winner. By Proposition 2, there is a simple rival of d'. Let d'' be a decision that maximizes $N_{d'}(\cdot)$. Since $P_S \cap (x_1, x_1') = \emptyset$, we may assume that $d''_L \cap [x_1, x_1') = \emptyset$. Then, we claim that $N_{d'}(d'') \leq N_d(d^*)$, where d^* is a decision modified from d''.

Since $\Delta(d, d') = \{1\}$, $N_{d'}(d'')|_i \neq N_d(d'')|_i$ implies $i = 0$ or $i = 1$. For $i = 0$ or $x_i' \notin d''_L$, $|\{x_i, x_{i+1}\} \cap d''_L| \leq 1$. In this case, let $d^* = d''$, and by Propositions 3 and 4 $N_{d'}(d'')|_i = N_d(d'')|_i$.

For $i = 1$ and $x_i' \in d''_L$, either $|\{x_i', x_{i+1}\} \cap d''_L| = 1$ or $|\{x_i', x_{i+1}\} \cap d''_L| = 2$. From d''_L, we replace x_1' with x_1, and let d^* be an envy-free decision with this set of facilities. By Propositions 4 and 5 it can be derived that $N_{d'}(d'')|_1 \leq N_d(d^*)|_1$.

Thus, the claim follows, and

$$0 < N_{d'}(d'') \leq N_d(d^*) \leq 0,$$

which is a contradiction. □

Remark 1. *Because of symmetry, x_k can be deployed at $p_{m_k^-}$.*

For $1 < h < k$, we show that x_h can be determined, depending on x_{h-1} and d_A. Let d' be a decision with $\Delta(d, d') = \{h\}$. Assume that $x_h' = b_h$, where the value b_h is to be determined. For a potential rival d'' of d' which maximizes $N_{d'}(\cdot)$, we show that d'' can be modified as a decision d^* so that $N_{d'}(d'') \leq N_d(d^*)$. Then d is a Condorcet winner implies that d' is also a Condorcet winner. It is clear that $N_{d'}(\cdot)|_i \leq N_d(d)|_i$ for $i \neq h-1$. Consider $N_{d'}(d^*)|_{h-1}$. By Propositions 3, 4, and 5, this partial margin depends on $f(x_{h-1}, x_h)$, $g^+(x_{h-1}, x_h)$, or $g^-(x_{h-1}, x_h)$. In the following, we show how Observation 2 enables us to ensure the property of having no simple rival.

4.1 Scoring functions with respect to a Condorcet winner

We intend to give an upper bound on a singular facility of a Condorcet winner, where the upper bound depends on the scoring functions and a predecessor. First, we show that in a Condorcet winner, $f(x_{h-1}, x_h)$ depends on x_{h-1} and d_A only.

Lemma 3. *Let $d = ((x_i, S_i))_{i=1}^k$ be a Condorcet winner of (S, k, P_S). For $1 < h < k$, if x_h is singular and $|S_h| < |S_{h-1}|$, then*

$$f(x_{h-1}, x_h) = n_{h-1}^+.$$

Proof. Clearly $f(x_{h-1}, x_h) \geq n_{h-1}^+$ since by moving x_{h-1} to $x_{h-1} + \epsilon$, there are n_{h-1}^+ voters prefer the newly deployed facility to the original one.

Suppose to the contrary that $f(x_{h-1}, x_h) > n_{h-1}^+$. If $|S_{h-1}|$ is odd, then $n_{h-1}^+ + 1 = (|S_{h-1}| + 1)/2$. By moving x_{h-1} and x_h towards right, a decision d' can be constructed with $N(d', d) \geq (n_{h-1}^+ + 1) + |S_h|/2 = (|S_{h-1}| + |S_h| + 1)/2$. If $|S_{h-1}|$ is even and $x_{h-1} = p_{m_{h-1}^-}$, then $n_{h-1}^+ + 1 = |S_{h-1}|/2 + 1$. By moving x_{h-1} and x_h towards right, a decision d' can be constructed with $N(d', d) \geq (n_{h-1}^+ + 1) + |S_h|/2 = (|S_{h-1}| + |S_h|)/2 + 1$. If $|S_{h-1}|$ is even and $x_{h-1} = p_{m_{h-1}^+}$, then $n_{h-1}^+ + 1 = |S_{h-1}|/2$. By moving x_{h-1} and x_h towards left, a decision d' can be constructed with $N(d', d) \geq |S_{h-1}|/2 + (n_{h-1}^+ + 1) = |S_{h-1}|$.

In all three cases, we have $N(d', d) + N(d, d') = |S_{h-1}| + |S_h|$ and $N(d', d) > (|S_{h-1}| + |S_h|)/2$. Hence, we know that $N(d', d) > N(d, d')$, which contradicts that d is a Condorcet winner. □

A similar argument as in the proof of Lemma 3 can be applied to derive $f(x_{h-1}, x_h)$ for $|S_h| = |S_{h-1}|$. The result is stated in Lemma 4.

Lemma 4. *Let $d = ((x_i, S_i))_{i=1}^k$ be a Condorcet winner of (S, k, P_S). For $1 < h < k$, if x_h is singular and $|S_h| = |S_{h-1}|$, then*

$$f(x_{h-1}, x_h) = |S_h|/2.$$

Proof. Omitted. □

By Lemmas 3 and 4, once x_{h-1} and d_A are given, the scoring function $f(x_{h-1}, x_h)$ can be determined. Recall from Observation 2 that x_h can be bounded above by given x_{h-1} and $f(x_{h-1}, x_h)$. When the regular decision under consideration is fixed, for $1 < h < k$ such that S_h is singular, we define

$$\tau(h) = \begin{cases} n_{h-1}^+, & \text{if } |S_h| < |S_{h-1}| \\ |S_h|/2, & \text{otherwise.} \end{cases}$$

In addition, let

$$\sigma_h = \min \{p_j - p_i \colon 1 \le i < j \le |S|, \{p_i, p_j\} \subseteq (x_{h-1}, p_{m_h^+}), j - i = \tau(h)\}.$$

Below we give upper bounds on x_h. The first two result from the property of having no simple rival.

Lemma 5. *Let $d = ((x_i, S_i))_{i=1}^k$ be a Condorcet winner of (S, k, P_S). For $1 < h < k$, we have*

$$(x_h - x_{h-1})/2 \le \sigma_h.$$

Proof. Omitted. □

Lemma 6. *Let $d = ((x_i, S_i))_{i=1}^k$ be a Condorcet winner of (S, k, P_S). For $1 < h < k$, if x_h is singular, then*

$$(x_h - x_{h-1})/2 \le p_{m_h^-} - p_{v_{h-1}^+}.$$

Proof. Suppose to the contrary that $p_{m_h^-} - p_{v_{h-1}^+} < (x_h - x_{h-1})/2$. Since $x_h > p_{m_h^-}$, we have $x_h^- = m_h^-$. It follows that $p_{x_h^-} - p_{v_{h-1}^+} < (x_h - x_{h-1})/2$, and by Lemma 1 there is a point y such that the $1 + n_h^-$ voters in $[p_{v_{h-1}^+}, p_{x_h^-}]$ prefer y to x_{h-1} and to x_h. Since $p_{m_h^-} < x_h$, we have $n_h^+ \le |S_h|/2 = n_h^-$. By moving x_h to y, we have a simple rival of d, which leads to a contradiction. □

The last bound on singular facility x_h results from the envy-freeness of a decision.

Lemma 7. *Let $d = ((x_i, S_i))_{i=1}^k$ be a decision of (S, k, P_S). If d is envy-free, then for $1 < i < k$*

$$x_i \le 2p_{v_i^-} - x_{i-1}.$$

By Lemmas 5, 6 and 7, for a Condorcet winner $d = ((x_i, S_i))_{i=1}^k$, if x_h is singular, then there is an upper bound b_h, derived as

$$b_h = \min \left\{ x_{h-1} + 2 \min \{ \sigma_h,\ p_{m_h^-} - p_{v_{h-1}^+} \},\ 2p_{v_h^-} - x_{h-1} \right\}. \quad (2)$$

4.2 A dominant decision

Given an instance (S, k, P_S), let $d = ((x_i, S_i))_{i=1}^k$ and $d' = ((x'_i, S_i))_{i=1}^k$ such that $\Delta(d, d') = \{h\}$. If S_h is singular and $x_h < x'_h \le \min\{b_h, p_{m_h^+}\}$, we claim that the existence of a simple rival of d' results in a simple rival of d. We assume that $|S_h| < |S_{h-1}|$, and leave the case $|S_h| = |S_{h-1}|$ to Appendix A.

Consider the scoring functions. By definition, we have

- $c(x'_{h-1}, x'_h) = c(x_{h-1}, x_h)$
- $c(x'_h, x'_{h+1}) \le c(x_h, x_{h+1})$
- $f(x'_h, x'_{h+1}) \le f(x_h, x_{h+1})$
- $g^+(x'_h, x'_{h+1}) \le g^+(x_h, x_{h+1})$
- $g^-(x'_h, x'_{h+1}) \le g^-(x_h, x_{h+1})$.

It remains to consider the relations between $f(x'_{h-1}, x'_h)$ and $f(x_{h-1}, x_h)$, $g^+(x'_{h-1}, x'_h)$ and $g^+(x_{h-1}, x_h)$, and $g^-(x'_{h-1}, x'_h)$ and $g^-(x_{h-1}, x_h)$.

Lemma 8. $f(x'_{h-1}, x'_h) = f(x_{h-1}, x_h)$.

Proof. By definition we have

$$f(x'_{h-1}, x'_h) \ge f(x_{h-1}, x_h).$$

To show that $f(x'_{h-1}, x'_h)$ is upper bounded by $f(x_{h-1}, x_h)$, recall the definition of x'_h. It can be derived that

$$(x'_h - x'_{h-1})/2 \le \sigma_h,$$

which implies

$$f(x'_{h-1}, x'_h) \le \tau(h).$$

Moreover, since x_h is a location of a singular facility, by Lemmas 3 and 4, we have

$$\tau(h) = f(x_{h-1}, x_h).$$

□

Lemma 9. $g^+(x'_{h-1}, x'_h) = g^+(x_{h-1}, x_h)$.

Proof. By definition, we have

$$n^+_{h-1} \le g^+(x_{h-1}, x_h) \le g^+(x'_{h-1}, x'_h) \le f(x'_{h-1}, x'_h),$$

and by Lemma 3, we have

$$f(x_{h-1}, x_h) = n^+_{h-1}.$$

Along with Lemma 8, the equalities hold. □

Lemma 10. $g^-(x'_{h-1}, x'_h) = g^-(x_{h-1}, x_h)$.

Proof. By definition,

$$f(x'_{h-1}, x'_h) \geq g^-(x'_{h-1}, x'_h) \geq g^-(x_{h-1}, x_h) \geq |S_h|/2.$$

Since d is regular and is a Condorcet winner,

$$f(x_{h-1}, x_h) \leq \lceil c(x_{h-1}, x_h)/2 \rceil \leq |S_h|/2 + 1.$$

Along with Lemma 8, we have $f(x'_{h-1}, x'_h) \leq |S_h|/2 + 1$. It follows that $g^-(x'_{h-1}, x'_h) = |S_h|/2+1$ only if $p_{m_h^-} - p_{v_{h-1}^+} < (x'_h - x'_{h-1})/2$. This implies that $p_{m_h^-} - p_{v_{h-1}^+} < (b_h - x'_{h-1})/2$, which is a contradiction. □

Remark 2. *For $|S_{h-1}| = |S_h|$, all inequalities mentioned above hold except that for g^+. It is possible that $g^+(x'_{h-1}, x'_h) = g^+(x_{h-1}, x_h) + 1$. For a simple rival d'' of d', if $N_{d'}(d'')|_{h-1} = g^+(x'_{h-1}, x'_h)$, we can modify d'' to be d^* so that $N_d(d^*)|_{h-1} \geq g^+(x'_{h-1}, x'_h)$. Details are given in Appendix A.*

Theorem 1. *Given an instance (S, k, P_S), let $d = ((x_i, S_i))_{i=1}^k$ and $d' = ((x'_i, S_i))_{i=1}^k$ be two regular decisions such that $\Delta(d, d') = \{h\}$. If S_h is singular and $p_{m_h^-} < x_h < x'_h \leq \min\{b_h, p_{m_h^+}\}$, then d is a Condorcet winner implies that d' is a Condorcet winner.*

Proof. (sketch) Suppose to the contrary that d is a Condorcet winner but d' is not. We may assume that d' has a simple rival because the envy-freeness follows from Lemma 7 and the envy-freeness of d. Let $d'' = ((x''_i, S''_i))_{i=1}^k$ be a simple rival of d' which maximizes $N_{d'}(\cdot)$. By Eq (1),

$$N_{d'}(d'') = \sum_{i=0}^{k} N_{d'}(d'')|_i - |P_S \cap (d'_L \setminus d''_L)|.$$

Let d^* be an envy-free decision such that

$$x^*_i = \begin{cases} x''_i, & \text{if } x''_i \neq x'_h \\ x_h, & \text{otherwise.} \end{cases}$$

If $|S_{h-1}| > |S_h|$, by Lemmas 5, 6, and 7, for $0 \leq i \leq k$ it can be derived from Propositions 3, 4, and 5 that

$$N_{d'}(d'')|_i \leq N_d(d^*)|_i$$

(with an exception indicated in Remark 3). In addition, $x_h \notin P_S$ implies $|P_S \cap (d_L \setminus d^*_L)| \leq |P_S \cap (d'_L \setminus d''_L)|$. It follows that

$$0 < \sum_{i=0}^{k} N_{d'}(d'')|_i - |P_S \cap (d'_L \setminus d''_L)| \leq \sum_{i=0}^{k} N_d(d^*)|_i - |P_S \cap (d_L \setminus d^*_L)| \leq 0,$$

which is a contradiction. For $|S_{h-1}| = |S_h|$, as noted in Remark 2, a contradiction can also be derived. □

Remark 3. *The strict inequality $c(x'_h, x'_{h+1}) < c(x_h, x_{h+1})$ implies $c(x'_h, x'_{h+1}) + 1 = c(x_h, x_{h+1})$. However, in this case $P_S \cap (d_L \setminus d^*_L)$ is a proper subset of $P_S \cap (d'_L \setminus d''_L)$.*

Theorem 1 leads to the following result.

Corollary 1. *Let d be a Condorcet winner of an instance (S, k, P_S). There is a Condorcet winner d' with $d'_A = d_A$ and $d'_L \subseteq \{p_{m_h^-}, p_{m_h^+}, b_h : h \in [k]\}$.*

For the example given in Figure 3, we let $x_2 = b_2 = \min\{15, 19\} = 15$. This decision is a Condorcet winner. Note that there are a right rival and a left rival for $x_2 = 12$ and $x_2 = 17$, respectively.

5 Algorithm

Based on Corollary 1, for an instance (S, k, P_S) one may implement the following procedure to determine the existence of a Condorcet winner.

1. Enumerate all k-partitions of a given instance.
2. For each k-partition, enumerate all deployments of facilities from

$$\{p_{m_h^-}\ ,\ p_{m_h^+}\ ,\ b_h\ :\ h \in [k]\}.$$

3. For a chosen decision, verify if it is a Condorcet winner.

For a Condorcet winner d, since d is regular, we have $||S_i| - |S_j|| \leq 2$ for $\{i, j\} \subseteq [k]$, and thus the number of k-partitions is of $O(3^k)$. Step 2 shows that the number of possible deployments of facilities is at most 3^k, given a k-partition. Let $T(n, k)$ be the time complexity for verifying if a decision is a Condorcet winner, where $n = |S|$. We have that a Condorcet winner can be computed in $O(3^{2k} \cdot T(n, k))$ time if it exists.

To verify if a decision $d = ((x_i, S_i))_{i=1}^k$ is a Condorcet winner, we propose an algorithm based on dynamic programming. The envy-freeness of a decision can easily be checked. To determine if there is a simple rival of decision d, we compute the maximum of $N_d(\cdot)$ recursively as follows. Because of symmetry, we show how $N_d(d')$ is computed for d' being a right rival of d.

Let $Margin(i, j)$ be the margin that is the optimum of

$$\begin{array}{ll} \text{maximize} & N_d(d') \\ \text{subject to} & d' \text{ is a right rival of } d \\ & \Delta(d, d') = \{i, i+1, \ldots, j\}. \end{array}$$

For $1 \leq i \leq m < j$ and $\ell \leq m - i + 1$, let $s(i, m, \ell, \text{ub})$ be the maximum results from deploying ℓ facilities in $(x_i, x_{m+1}]$, with the restriction that one of the facilities coincides with x_{m+1} if $\text{ub} = \text{TRUE}$. Let

$$\delta_i = \begin{cases} 1, & \text{if } x_i \in P_S \\ 0, & \text{otherwise.} \end{cases}$$

By Propositions 3, 4 and 5, we have the following recursive formulae.

$$\begin{aligned} s(i, m, \ell, \text{TRUE}) = \max\{ & s(i, m-1, \ell-1, \text{FALSE}) - n_m^+, \\ & s(i, m-1, \ell-2, \text{FALSE}) + g^+(x_m, x_{m+1}), \\ & s(i, m-1, \ell-3, \text{FALSE}) + c(x_m, x_{m+1}), \\ & s(i, m-1, \ell-1, \text{TRUE}), \\ & s(i, m-1, \ell-2, \text{TRUE}) + f(x_m, x_{m+1}), \\ & s(i, m-1, \ell-3, \text{TRUE}) + c(x_m, x_{m+1})\}. \end{aligned}$$

$$\begin{aligned} s(i, m, \ell, \text{FALSE}) = \max\{ & s(i, m-1, \ell, \text{FALSE}) - c(x_m, x_{m+1}) - \delta_{m+1}, \\ & s(i, m-1, \ell-1, \text{FALSE}) + 2f(x_m, x_{m+1}) - c(x_m, x_{m+1}) - \delta_{m+1}, \\ & s(i, m-1, \ell-2, \text{FALSE}) + c(x_m, x_{m+1}) - \delta_{m+1}, \\ & s(i, m-1, \ell, \text{TRUE}) - n_{m+1}^- - \delta_{m+1}, \\ & s(i, m-1, \ell-1, \text{TRUE}) + g^-(x_m, x_{m+1}) - \delta_{m+1}, \\ & s(i, m-1, \ell-2, \text{TRUE}) + c(x_m, x_{m+1}) - \delta_{m+1}\}. \end{aligned}$$

If $i < j < k$,

$$
\begin{aligned}
Margin(i,j) = \max\{ & s(i, j-1, j-i, \text{FALSE}) + g^{+}(x_j, x_{j+1}), \\
& s(i, j-1, j-i-1, \text{FALSE}) + c(x_j, x_{j+1}), \\
& s(i, j-1, j-i, \text{TRUE}) + f(x_j, x_{j+1}), \\
& s(i, j-1, j-i-1, \text{TRUE}) + c(x_j, x_{j+1})\}.
\end{aligned}
$$

If $i < j = k$,

$$
\begin{aligned}
Margin(i,j) = \max\{ & s(i, j-1, j-i, \text{FALSE}) + n_j^{+}, \\
& s(i, j-1, j-i, \text{TRUE}) + n_j^{+}\}.
\end{aligned}
$$

The terminal conditions hold when $\ell = 0$ or $i = m$, namely

$$
s(i, m, 0, \text{ub}) = \begin{cases} -\sum_{y=i}^{m} |S_y| - n_{m+1}^{-} - \delta_{m+1}, & \text{if ub} = \text{FALSE} \\ -\infty, & \text{if ub} = \text{TRUE} \end{cases}
$$

$$
s(i, i, \ell, \text{ub}) = \begin{cases} -n_i^{-} + 2f(x_i, x_{i+1}) - c(x_i, x_{i+1}) - \delta_i - \delta_{i+1}, & \text{if } \ell = 1 \text{ and ub} = \text{FALSE} \\ -|S_i|, & \text{if } \ell = 1 \text{ and ub} = \text{TRUE} \\ -\infty, & \text{if } \ell \geq 2. \end{cases}
$$

$$
Margin(i,j) = \begin{cases} -n_i^{-} + g^{+}(x_i, x_{i+1}) - \delta_i, & \text{if } i = j < k \\ -n_i^{-} + n_i^{+} - \delta_i, & \text{if } i = j = k. \end{cases}
$$

Remark 4. *By reversing the x-axis, the recursive formulae given above are applied to derive $N_d(d')$ for d' being a left rival of d. For convenience, we use $Margin'$ and s' to differentiate.*

Decision d has a simple rival if and only if

$$
\max_{1 \leq i \leq j \leq k} Margin(i,j) > 0 \quad \text{or} \quad \max_{1 \leq i \leq j \leq k} Margin'(i,j) > 0.
$$

For $0 \leq i \leq k$, the values n_i^{-}, n_i^{+}, $f(x_i, x_{i+1})$, $c(x_i, x_{i+1})$, $g^{+}(x_i, x_{i+1})$, and $g^{-}(x_i, x_{i+1})$ can be computed in $O(n)$ time. With this preprocessing, the computation can be done in $O(k^3)$ time, using dynamic programming. Thus, $T(n,k) \in O(n + k^3)$.

Theorem 2. *Given an instance (S, k, P_S), determining whether a Condorcet winner exists takes $O(3^{2k}(n + k^3))$ time, where $n = |S|$. Moreover, a Condorcet winner can be computed if it exists.*

Note that the number of k-partitions is not of $\Omega(3^{2k})$, as to $k = n$ the n-partition is unique.

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A Scoring functions for $|S_h| = |S_{h-1}|$

Consider two regular decisions $d = ((x_i, S_i))_{i=1}^k$ and $d' = ((x'_i, S_i))_{i=1}^k$. Assume that $\Delta(d, d') = \{h\}$, S_h is singular and $p_{m_h^-} < x_h < x'_h \leq \min\{b_h, p_{m_h^+}\}$.

Lemma 11. *If $|S_{h-1}| = |S_h|$ and d'' is a decision such that $N_{d'}(d'') > 0$ and $N_{d'}(d'')|_{h-1} = g^+(x'_{h-1}, x'_h)$, then*

$$g^+(x'_{h-1}, x'_h) = g^+(x_{h-1}, x_h) + 1 \implies d \text{ is not a Condorcet winner.}$$

Proof. Since $|S_{h-1}| = |S_h|$ by definition

$$\frac{|S_h|}{2} - 1 \leq g^+(x_{h-1}, x_h) \leq g^+(x'_{h-1}, x'_h) \leq f(x'_{h-1}, x'_h) \leq \frac{|S_h|}{2}.$$

The assumption $g^+(x'_{h-1}, x'_h) = g^+(x_{h-1}, x_h) + 1$ implies

$$g^+(x_{h-1}, x_h) = \frac{|S_h|}{2} - 1. \tag{3}$$

and

$$g^+(x'_{h-1}, x'_h) = \frac{|S_h|}{2}.$$

Moreover, Eq. (3) holds only if $x_{h-1} = p_{m_{h-1}^+}$, which implies

$$c(x_{h-1}, x_h) = |S_h| - 1.$$

Along with Lemma 8, we have

$$f(x_{h-1}, x_h) = f(x'_{h-1}, x'_h) = \frac{|S_h|}{2}.$$

We claim that there is a decision d^* such that $N_d(d^*) > 0$. Since $N_{d'}(d'')|_{h-1} = g^+(x'_{h-1}, x'_h)$, we may assume that there is exactly one facility x''_j belonging to (x'_{h-1}, x'_h), and x''_{j+1} coincides with x'_h. If $|\{i \in S_h\colon\ x''_j \prec_i x''_{j+1}\}| = 0$, then let d^* be a decision modified from d'' by moving x''_j towards right properly. The difference on the margin satisfies

$$N_d(d^*) - N_{d'}(d'') = 2f(x_{h-1}, x_h) - c(x_{h-1}, x_h) + \frac{|S_h|}{2} - g^+(x'_{h-1}, x'_h) > 0.$$

Otherwise, move both x''_j and x''_{j+1} into (x_{h-1}, x_h), and it follows that

$$N_d(d^*) - N_{d'}(d'') \geq c(x_{h-1}, x_h) - \left(\frac{|S_h|}{2} - 1\right) - g^+(x'_{h-1}, x'_h) = 0.$$

In either case, the difference is nonnegative, and the claim follows. □

B A remark on Hajduková's algorithm

To verify if a given decision is a Condorcet winner, Hajduková [11] developed an algorithm, where the envy-freeness and the existence of a simple rival are verified. In Hajduková's algorithm, the existence of a (right) simple rival is affirmed if one of the following holds: for $1 \leq i \leq j \leq k$

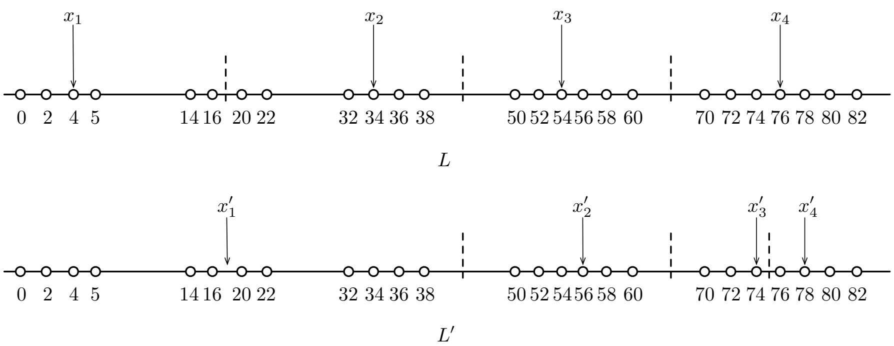
$$\sum_{h=i}^{j-1} f(x_h, x_{h+1}) + n_j^+ > \frac{1}{2} \sum_{h=i}^{j} |S_h|.$$

$$\sum_{h=i}^{j-1} f(x_h, x_{h+1}) + n_j^+ = \frac{1}{2} \sum_{h=i}^{j} |S_h| \quad \text{and} \quad p_{v_{j+1}^-} - p_{x_j^+} < (x_{j+1} - x_j)/2.$$

Nevertheless, the verification works correctly if and only if

$$d' \text{ is a simple rival of } d \text{ with } \Delta(d', d) = \{h \in [n] \colon\ i \leq h \leq j\} \implies \text{for } i \leq h \leq j,\ x'_h \in (x_h, x_{h+1}).$$

Notice that the statement is not true, while the following is a counterexample.



The two figures demonstrate two decisions of an instance with $k = 4$. The upper one, say d, is regular, envy-free, and supposed to have no simple rival according to Hajduková's algorithm. However, the lower one, with $x'_2 \in (x_3, x_4)$, is a simple rival of d.

C An algorithm for computing the votes of a potential rival

For a decision d, here we present in Algorithm 1 how $f(x_h, x_{h+1})$, $g^+(x_h, x_{h+1})$ and $g^-(x_h, x_{h+1})$ are computed. The values n_h^-, n_h^+, and $c(x_h, x_{h+1})$ can be computed in $O(n)$ time straightforwardly.

Algorithm 1: Computing $f(x_h, x_{h+1})$, $g^+(x_h, x_{h+1})$ and $g^-(x_h, x_{h+1})$

Input: x_h, x_{h+1}, voters located in (x_h, x_{h+1})
Output: $f(x_h, x_{h+1})$, $g^+(x_h, x_{h+1})$ and $g^-(x_h, x_{h+1})$

begin
    $f \longleftarrow 0$
    $g^+ \longleftarrow 0$
    $g^- \longleftarrow 0$
    $\mathrm{ctr} \longleftarrow 0$
    $i \longleftarrow x_h^+$
    $j \longleftarrow i$
    while $p_j < x_{h+1}$ **do**
        $\mathrm{ctr} \longleftarrow \mathrm{ctr} + 1$
        while $p_j - p_i \geq (x_{h+1} - x_h)/2$ **do**
            $i \longleftarrow i + 1$
            $\mathrm{ctr} \longleftarrow \mathrm{ctr} - 1$
        $f \longleftarrow \max\{\, f, \mathrm{ctr} \,\}$
        $g^+ \longleftarrow \max\{\, g^+, \mathrm{ctr} - i + x_h^+ \,\}$
        $g^- \longleftarrow \max\{\, g^-, \mathrm{ctr} + j - x_{h+1}^- \,\}$
        $j \longleftarrow j + 1$
    return f, g^+, g^-

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